# Exercise Sheet 3

November 11, 2012

1. Where are the following functions defined:

   (a) $\frac{1}{\exp(z)}$

   (b) $\frac{1}{\exp(z)-1}$

   (c) $\frac{1}{\exp(z)-i}$

   (d) $\frac{1}{\sin(z)}$

2. The exponential function is defined by:

$$\exp(x+iy) = \exp(x)(\cos(y) + i\sin(y))$$

   Using CR equations prove that it is complex differentiable.

3. Describe the image of the following sets under the exponential map:

   (a) The real line.

   (b) Lines parallel to the real line.

   (c) Lines perpendicular to the real line.

   (d) The line given by $\{z \mid Re(z) = Im(z)\}$.

4. Prove the identity $\cos^2(z) + \sin^2(z) = 1$ for complex numbers.

5. Compute:

   (a) $\displaystyle\int_{|z|=1} \frac{dz}{z}$

   (b) $\displaystyle\int_{|z|=1} \frac{dz}{z^2}$

6. Let $f : \mathbb{C} \to \mathbb{C}$ be a bounded function.

   (a) Prove: $\displaystyle\lim_{R\to 0} \int_{|z|=R} f(z)dz = 0$.

   (b) Is $\displaystyle\lim_{R\to\infty} \int_{|z|=R} f(z)dz = \infty$? prove or provide a counterexample.

7. Let $g(z) = \sum a_n z^n$ be a power series such that:

$$\limsup \sqrt[n]{|a_n|} < 1$$

Prove:

$$\int_{|z|=1} g(z)dz = 0$$

8. Compute the radius of convergence for the following power series and sketch the circle of convergence:

(a) $\sum (2 + (-1)^n)^n \cdot z^n$

(b) $\sum 2^n \cdot (z - i)^n$

(c) $\sum n! \cdot (z - (1 + i))^n$

(d) $\sum (z + 3)^{n^2}$

9. Let f be defined by $f(z) = \frac{\exp(iz)}{z}$, let R be a positive number and let $\gamma_R$ be the path which goes from $R$ to $R + iR$ to $-R + iR$ and than to $-R$ in straight lines. Prove:

$$\int_{\gamma_R} f(z)dz \overset{R\to\infty}{\longrightarrow} 0$$

Hint: Divide $\gamma_R$ to 3 parts and apply a bound for each one of them.
Hint 2: The divison is not to three straight lines, it is more sophisticated then that.